Spatial Correlation Method and a Time-Varying Flexible Structure

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A new "spatial correlation method" is employed to calculate the second-order statistics of the response of lifting rotor blades with negligible bending stiffness in forward flight to excitations which are random functions of both space and time. Results obtained herein together with the rigid flapping solution delimit the range of the meansquare properties of flexible blade response. The level of computation involved in obtaining these results supports our theoretical estimate which shows that a numerical solution of the meansquare properties of complex random response processes by the spatial correlation method is several hundred folds more efficient than any other method available in the literature. Further improvements in the computational efficiency of the method, not noted previously, are shown to be possible for specific classes of problems.

Nomenclature

l	= beam length
EI	= beam bending stiffness
Ω	= constant angular velocity of the rotating beam
Z	= axis of rotation through the hinged end of the beam and normal to the rotor plane (the X, Y-plane)
m	= linear mass density (mass per unit length) of the beam
w	= dimensionless transverse displacement of the beam with $l \cdot w$ being the actual displacement
τ	= dimensionless time (with the free end of the beam pointing in the direction opposite to the direction of V_f at $\tau = 0$), τ/Ω being the real time
x	= dimensionless distance with $l \cdot x$ being the distance along the beam span measured from the hinged end
V_f	= the constant forward speed of the rotating beam system
μ	= the advance ratio ($= V_f/\Omega l$)
γ_0	= $\gamma = 6\gamma_0 = \rho cal^2/m$ is the Lock number
ρ	= air density
c	= blade chord
a	= lift curve slope
$n(x,\tau)$	= inflow ratio [see Eq. (2) and discussion that follows]
u, s, t, v	= spatial correlation functions [see Eq. (4)]
$r(x_1, \tau_1; x_2, \tau_2)$	= the autocorrelation function of the blade displacement ($= \langle w(x_1,\tau_1)w(x_2,\tau_2)\rangle$)
$\langle \dots \rangle$	= ensemble average
$L_{x\tau}[\ ]$	= a linear differential operator defined by Eq. (6)
$\hat{w}$	= the matrix of SCF [see Eq. (12)]
SCF	= spatial correlation function(s)
p, q	= load-response spatial correlation functions [see Eq. (22)]
$R_S(x_1, x_2, \tau)$	= see Eqs. (14) and (17)
Δ, h	= time step and mesh spacing
ε	= ε^{-1} is a dimensionless correlation length [see Eqs. (16) and (29)]
α	= α^{-1} is a dimensionless correlation time [see Eq. (29) and the discussion followed]
$\delta(\tau), H(\tau)$	= Dirac delta function and Heaviside unit step-function
$\hat{n}(x,\tau)$	= an auxiliary zero mean, temporally uncorrelated random function [see Eq. (18)]
$c(x,\tau; x',\tau')$	= load-response correlation ($= \langle n(x,\tau)w(x',\tau')\rangle$)
$U^k_{m,n}, \dots, Q^k_{m,n}$	= the value of $u, \dots, q$ at $x_m = mh$, $y_n = nh$ and $\tau_k = k\Delta$

1. Introduction

CONSIDER a uniform flexible beam of length l and bending stiffness EI, hinged at one end and free at the other end. The beam is rotating with a constant angular velocity Ω about an axis (the Z-axis) through the hinged end. At any instance, the beam lies somewhere in a rotor-plane (the X, Y-plane) and the Z-axis is normal to this plane (Fig. 1). We are interested here in the out-of-rotor-plane flexural motion of the rotating beam when it is subject to some external random excitation. For small amplitude motion, the dynamic of the rotating beam is adequately described by the equation

$$m\Omega^2\{w_{\tau\tau} - [\tfrac{1}{2}(1-x^2)w_x]_x\} + (EI/l^4)w_{xxxx} = f(x,\tau) \qquad (0 < x < 1, \tau > 0) \qquad (1)$$

where m is the linear mass density of the beam, $l \cdot w(x,\tau)$ is the transverse displacement (in the direction of the Z-axis), τ/Ω is the real time, and $l \cdot x$ is the distance along the beam span measured from the hinged end. If the rotating beam system is also moving in the atmosphere with a constant forward speed V_f in some fixed direction in the rotor-plane, the beam experiences a time-varying aerodynamic lift force[1]

$$f(x,\tau) = m\Omega^2\gamma_0|x + \mu\sin\tau|[n(x,\tau) - w_\tau - \mu\cos\tau\, w_x] \qquad (2)$$

where, in the terminology of the rotor blade literatures, $\mu = V_f/\Omega l$ is the advance ratio, $\gamma = 6\gamma_0 = \rho cal^2/m$ is the Lock number (see also the nomenclature) and $n(x,\tau)$ is the inflow ratio, the ratio of transverse airflow velocity to rotating speed at the free end, Ωl. We assume that the rotating beam experiences no transverse motion prior to some reference time $\tau = 0$ so that

$$w(x,0) = w_\tau(x,0) = 0 \qquad (0 \leq x \leq 1) \qquad (3)$$

Along with suitable end conditions at $x = 0$ and $x = 1$, Eqs. (1–3) define a linear initial-value problem which is appropriate for the analysis of the small amplitude transverse motion of flexible lifting rotor blades. We are concerned here with the blade response when the inflow ratio, $n(x,\tau)$, is a random

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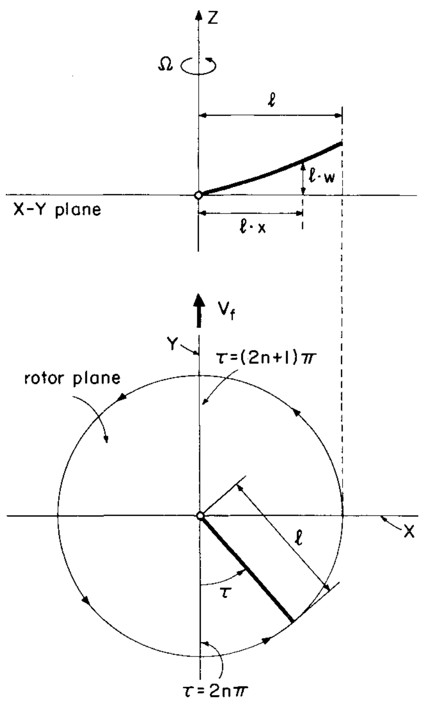

Fig. 1 A flexible rotor blade.

function with known statistics. For such a linear initial value problem with random forcing, there are several methods available for calculating the response statistics.[2-4] Unfortunately, because of the time-varying blade properties, the steady-state response of the blade will be nonstationary even if the random forcing $n(x, \tau)$ is a stationary process. Just to compute the steady-state meansquare nonstationary response for a typical stochastic loading by any of the conventional methods would take over a day of machine calculation.[5] To render this difficult problem more tractable, the pioneering works in the area of lifting rotors excited by random loads consider only rigid blades with excitations uniform along the blade span.[6-9] More recent investigations[10-12] allow for blade flexibility but continue to treat only time dependent excitations. On the other hand, one of the main sources of random excitation on a lifting rotor is the rotor-generated turbulence. While very little quantitative information about this random process is available, the corresponding random loading on the rotor blade is very likely to vary along the blade span with a correlation length less than or equal to the blade length. In that case, the simplifying assumption of a spatially uniform excitation often does not lead to an adequate solution for the blade response as demonstrated in Ref. 13. A flexible blade analysis which allows for an assessment of the effect of a space-time random loading is clearly warranted. To be useful, such an analysis should require only a "reasonable" amount of machine computation. From this point of view, the Green's function method and the autocorrelation method[2-4] though applicable, are impractical.[5,14]

To efficiently handle the rotor blade problem described previously and other problems associated with complex flexible structures excited by random load, the first author developed in Refs. 5 and 14 a new method to obtain the (first- and) second-order response statistics for general linear dynamical problems of deformable bodies. The random excitation involved may be temporally nonstationary and spatially inhomogeneous. The essential feature of this new method is the formulation of a (nonstochastic) initial value problem for the *spatial correlation function*(s) (SCF) of the response process. For the rotor blade problem, we shall see in the next section that the relevant SCF are

$$u(x, y, \tau) = \langle w(x, \tau) w(y, \tau) \rangle, \quad s(x, y, \tau) = \langle w(x, \tau) w_\tau(y, \tau) \rangle$$
$$t(x, y, \tau) = \langle w_\tau(x, \tau) w(y, \tau) \rangle, \quad v(x, y, \tau) = \langle w_\tau(x, \tau) w_\tau(y, \tau) \rangle \quad (4)$$

where $\langle \ldots \rangle$ is the ensemble-averaging operation and $0 \leqq x, y \leqq 1$. On the one hand, these SCF contain the meansquare response as a special case (with $y = x$). On the other hand, they serve as the initial conditions for another (nonstochastic) initial-value problem for the determination of the autocorrelation function of the blade response, $r(x_1, \tau_1; x_2, \tau_2) \equiv \langle w(x_1, \tau_1) w(x_2, \tau_2) \rangle$.

The present paper reports the first serious application of this new method. Our purpose here is to demonstrate the actual working of the general method developed in Refs. 5 and 14 and to investigate the effect of a spanwise load correlation on flexible lifting rotor blades. For these purposes, we will limit ourselves to a study of the extreme case of flexible blades with no bending stiffness ($EI = 0$) by the spatial correlation method. The results for this "string" model of rotor blades and those obtained in Ref. 13 for rigid blades delimit the range of the response statistics for flexible blades with nonvanishing bending stiffness. We note in passing that the effective bending stiffness factor $\zeta^4 \equiv EI/\Omega^2 l^4 m$ is around 0.06 for many existing blades.

As far as the actual working of the method is concerned, we found that the spatial correlation method requires several hundred folds less machine time to determine the steady-state meansquare response of the blade than the two other methods mentioned earlier (see Ref. 5 and also Sec. 3 and Appendix A herein). For the simpler case of hovering and vertical flight for which $\mu = 0$, it is possible to improve upon the computational aspect of the general method. This improvement along with an investigation of the effect of the blade bending stiffness has been reported in Ref. 15, so that we can focus our attention on the more difficult case of forward flight ($\mu > 0$), though the computer code developed can be used for the $\mu = 0$ case as well.

2. Spatial Correlation Functions

For a blade with no bending stiffness, the equation of motion, Eq. (1), [with $f(x, \tau)$ given by Eq. (2)] can be written as

$$w_{\tau\tau} + \gamma_0 |x + \mu \sin \tau| w_\tau - L_{x\tau}[w] = \gamma_0 |x + \mu \sin \tau| n(x, \tau)$$
$$(0 < x < 1, \tau > 0) \quad (5)$$

where

$$L_{x\tau}[\ ] = \tfrac{1}{2}(1 - x^2)[\ ]_{xx} - (x + \gamma_0 \mu \cos \tau |x + \mu \sin \tau|)[\ ]_x \quad (6)$$

The lifting rotor "string" is fixed at $x = 0$ and free at $x = 1$ so that

$$w(0, \tau) = \lim_{x \to 1} [(1 - x^2) w_x] = 0 \qquad (\tau \geqq 0) \quad (7)$$

The second condition in Eq. (7) is satisfied if w_x remains bounded at $x = 1$.

We are interested here in the solution of Eqs. (3) and (5–7) when $n(x, \tau)$ is a zero mean random function with known statistics. Since the problem is linear, the response process $w(x, \tau)$ will also be of zero mean and we can concentrate on the second-order statistics of $w(x, \tau)$ characterized by the autocorrelation function $r(x_1, \tau_1; x_2, \tau_2)$ defined earlier. To determine the autocorrelation of $w(x, \tau)$ by the spatial correlation method developed in Refs. 5 and 14, we first obtain the four spatial correlation functions u, s, t, and v defined by Eq. (4). To do so, we observe that

$$u_\tau = \langle w_\tau(x, \tau) w(y, \tau) \rangle + \langle w(x, \tau) w_\tau(y, \tau) \rangle = t + s \quad (8)$$

and

$$t_\tau = v(x, y, \tau) + \langle w_{\tau\tau}(x, \tau) w(y, \tau) \rangle$$

where we have made use of the fact that, within the framework of meansquare convergence, differentiation commutes with the ensemble averaging operation.[2] If Eq. (5) is now used to eliminate $w_{\tau\tau}$ from the expression for t_τ, we get

$$t_\tau = v + L_{x\tau}[u] - \gamma_0 |x + \mu \sin \tau| t + \gamma_0 |x + \mu \sin \tau| p(x, y, \tau) \quad (9)$$

where $p(x, y, \tau) = \langle n(x, \tau) w(y, \tau) \rangle$. Interchange the role of x and y and we have also

$$s_\tau = v + L_{y\tau}[u] - \gamma_0 |y + \mu \sin \tau| s + \gamma_0 |y + \mu \sin \tau| p(y, x, \tau) \quad (10)$$

Finally, similar manipulations applied to the expression for v_τ give

$$v_\tau = L_{x\tau}[s]+L_{y\tau}[t]-\gamma_0(|x+\mu\sin\tau|+|y+\mu\sin\tau|)v+g(x,y,\tau) \quad (11)$$

where

$$g(x,y,\tau) = \gamma_0|x+\mu\sin\tau|q(x,y,\tau)+\gamma_0|y+\mu\sin\tau|q(y,x,\tau)$$

with

$$q(x,y,\tau) = \langle n(x,\tau)w_\tau(y,\tau)\rangle$$

Equations (8–11) are to be satisfied in the interior of the semi-infinite unit square column $(0<x,y<1,\tau>0)$ in the x, y, τ-space. On the base square of the column, $\tau=0$, we have from the initial conditions (3)

$$\hat{w}(x,y,0) \equiv \begin{bmatrix} u & s \\ t & v \end{bmatrix}_{\tau=0} = 0 \qquad (0 \leqq x,y \leqq 1) \quad (12)$$

For $\tau > 0$, we have from the boundary conditions (7)

$$\hat{w}(0,y,\tau) = \hat{w}(x,0,\tau) = 0$$
$$\lim_{x\to 1}[(1-x^2)\hat{w}_x] = \lim_{y\to 1}[(1-y^2)\hat{w}_y] = 0 \quad (13)$$

as the boundary conditions on the four walls of the square column. We note that some of the conditions in Eq. (13) are redundant, but are included to take advantage of the compact matrix notation.

Now, if we can somehow determine the two yet unknown functions $p(x,y,\tau)$ and $q(x,y,\tau)$ (which involve the unknown w), the four PDE Eqs. (8–11), the initial conditions Eq. (12), and the boundary conditions Eq. (13) completely determine the four SCF. The meansquare response of the blade can then be obtained from these SCF by simply setting $y=x$. In this paper, we will discuss the solution for p and q for two classes of random excitations which are pertinent to the rotor blade problem. The results are then used in Eqs. (9–11) for the solution of the SCF, u, s, t, and v. A method for the determination of p and q for more general $n(x,\tau)$ can be found in Refs. 5 and 14.

Having the SCF, u, s, t, and v, we can then obtain, if we wish, the autocorrelation function r of the response $w(x,\tau)$ by a method outlined in Appendix A.

3. Temporally Uncorrelated Excitations

We consider in this section a general class of zero mean random inflow ratios, $n(x,\tau)$, with

$$\langle n(x_1,\tau_1)n(x_2,\tau_2)\rangle = R_S(x_1,x_2,\tau_1)\delta(\tau_2-\tau_1) \quad (14)$$

where $R_S(x_2,x_1,\tau_1)=R_S(x_1,x_2,\tau_1)$. For a fixed point in space, the excitation is simply a (nonstationary) shot noise process. For this class of temporally uncorrelated excitations, it can be shown[5,14] that

$$p(x,y,\tau)\equiv 0, \quad q(x,y,\tau)=\tfrac{1}{2}\gamma_0|y+\mu\sin\tau|R_S(x,y,\tau)$$
$$g(x,y,\tau)=\gamma_0{}^2|x+\mu\sin\tau||y+\mu\sin\tau|R_S(x,y,\tau) \quad (15)$$

With $p(x,y,\tau)$ and $q(x,y,\tau)$ completely determined, the initial-value problem defined by Eqs. (8–13), may be solved for the four SCF. In principle, a numerical solution of this problem can be obtained by a straightforward explicit finite difference scheme using a first-order forward difference formula for the time derivatives (with a time step Δ) and second-order central difference formulas for the spatial derivatives (with the same mesh spacing h in both x and y). In view of the apparent parabolic form of the PDE, Eqs. (8–11), it would seem that for such an explicit scheme, stability of the numerical solution may be achieved only if $\Delta = O(h^2)$. To allow for a larger time step (and hopefully thereby to speed up the solution process), we would have to go to an implicit scheme which, in view of the time-varying coefficients, complicates the solution process significantly. Our experiments with different time steps for the straightforward explicit scheme show that numerical instability does occur if we use $\Delta = O(h)$.

One of the contributions of this paper is the development of a modified *explicit* scheme for a finite-difference solution of Eqs. (8–13) which allows the use of $\Delta = O(h)$ (see Appendix B). The modified explicit scheme is used for all the calculations done in connection with this paper and is stable in all cases. For a 21×21 grid with equal spacing over the unit square ($h=0.05$) and a time step $\Delta=\pi/80$, it takes about 5 min on a UNIVAC 1106 to generate the SCF for all grid points over five blade revolutions ($0\leqq\tau\leqq 10\pi$) using only single precision arithmetic. For $\gamma \geqq 4$, the results for the fourth and fifth revolution agree up to five significant figures so that the transient effect is negligible and a steady state solution (periodic in time) is obtained. The same results by the autocorrelation method described in Appendix A would have required over a day of calculation on the same machine.

It may be appropriate to mention at this point that as long as a numerical solution is necessary, we could have discretized the original problem Eqs. (3) and (5–7) in space and then calculated the covariance matrix and the correlation matrix for the resulting time varying discrete multidegree freedom dynamical system excited by random noise in a manner similar to what was done recently in Refs. 8 and 9 for a rigid blade. However, the foregoing discussion and the results of Appendix B leave no doubt that a numerical solution via the SCF method allows us to readily assess the stability of the numerical scheme employed. It is not likely that we would have arrived at the modified explicit scheme (which drastically reduces the computing time) had we discretized the continuous problem Eqs. (3) and (5–7) ab initio.

4. Steady-State Meansquare Response to Temporally Uncorrelated Excitation

The solution scheme developed in Sec. 3 and Appendix B was used to calculate the four spatial correlation functions of the rotor string problem [modelled by Eqs. (3) and (5–7)] for the case

$$\langle n(x_1,\tau_1)n(x_2,\tau_2)\rangle = e^{-\varepsilon|x_1-x_2|}\delta(\tau_2-\tau_1) \qquad (0\leqq x_1,x_2\leqq 1) \quad (16)$$

Some of the results of our calculations will be discussed in this section. We are particularly interested in: 1) the difference between the steady-state meansquare response of the two extreme cases, a string and a rigid blade,‡ to a spatially uniform inflow ($\varepsilon=0$); and 2) the effect of the dimensionless correlation length ε^{-1} which characterizes our particular spatially varying inflow. Note that for a fixed value of x, the steady-state meansquare properties of the string response are periodic functions of τ with a period 2π.

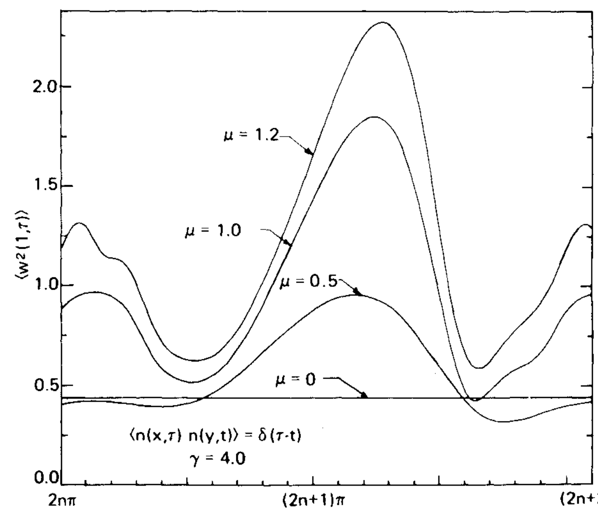

Fig. 2 Steady-state meansquare displacement at the blade tip for a spatially uniform and temporally uncorrelated inflow.

‡ Unless specified otherwise, the rigid blade is hinged at the axis of rotation.

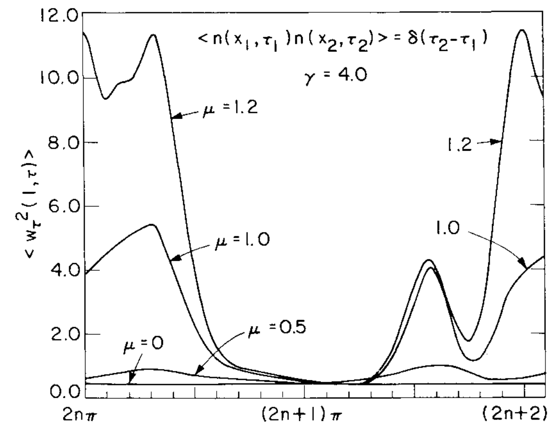

Fig. 3 Steady-state meansquare velocity at the blade tip for a spatially uniform and temporally uncorrelated inflow.

In Fig. 2, we show the steady-state meansquare displacement at the tip, $x = 1$, as a function of τ for $\varepsilon = 0$, $\gamma = 4$ and $\mu = 0.5$, 1.0, and 1.2. The results are qualitatively very similar to the corresponding rigid blade solution of Ref. 8. For example, the general shape of the curve is the same and the location and the magnitude of the peak value are nearly the same. The less than 10% discrepancy in the peak value is probably due in part to a tip loss factor of 0.97 incorporated in the analysis of Ref. 8 but not in our calculation. The pointwise agreement for each curve is better for $\mu = 0.5$ than for $\mu = 1.2$. For the latter case, our curve exhibits a secondary peak near the beginning of the forward stroke much larger in magnitude than the corresponding value for a rigid blade. The minimum tip meansquare displacement of our rotor string is also larger than that of a rigid blade.

Figure 3 shows the corresponding meansquare velocity. Its peak value increases much more rapidly with μ than the tip meansquare displacement. For $\mu = 0.5$, there are two peaks of comparable magnitude, one near the beginning of the forward stroke of the blade and the other near the end of the backward stroke. As μ increases, the first peak becomes more and more dominant. For the $\mu = 1.2$ case, a third peak, comparable to the

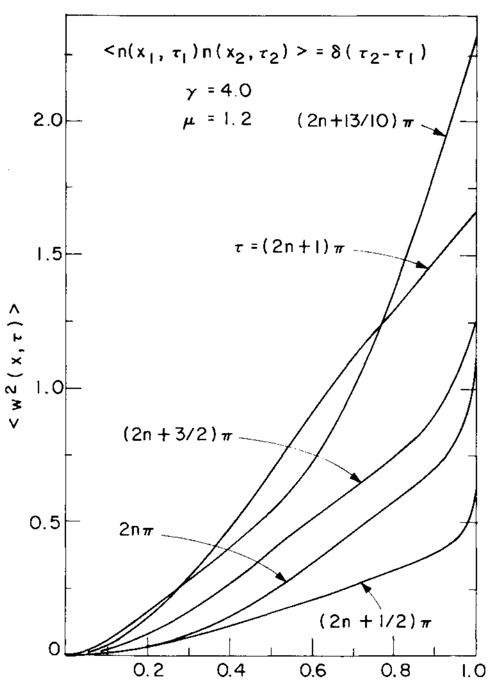

Fig. 4 Distributions of meansquare displacement along the blade span for a spatially uniform and temporally uncorrelated inflow.

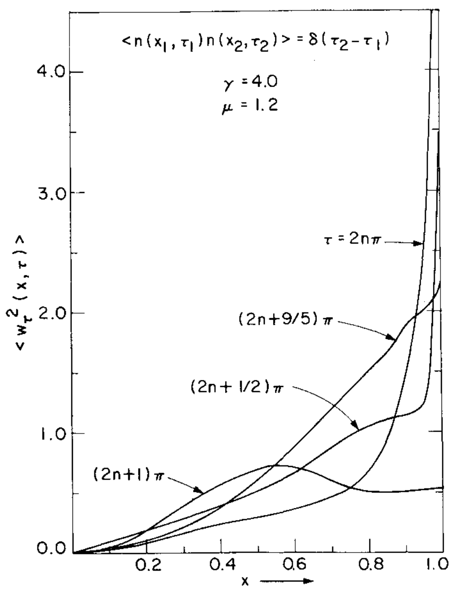

Fig. 5 Distributions of meansquare velocity along the blade span for a spatially uniform and temporally uncorrelated inflow.

first in magnitude appears at the beginning of the forward stroke.

In Fig. 4, we plotted the distribution of the steady-state meansquare displacement along the string for some typical values of τ (with $\gamma = 4$ and $\mu = 1.2$). It is rather evident from these plots that the dominant motion of the string is not nearly a rigid flapping motion. However, in all cases examined, $\langle w^2(x, \tau_{max})\rangle$ does vary almost as x^2 along the blade at the instant τ_{max} when the peak meansquare tip displacement is attained $\{\tau_{max} \approx [2n+(13/10)]\pi$ for the case plotted in Fig. 4$\}$.

Figure 5 shows the corresponding distributions of $\langle w_\tau{}^2(x,\tau)\rangle$. In contrast to the distributions of $\langle w^2(x,\tau)\rangle$, the maximum value of the steady-state meansquare velocity for a fixed τ does not always appear at the tip $x = 1$. For example, the spatial variation of $\langle w_\tau{}^2(x,\tau)\rangle$ at the end of the forward stroke, $\tau = (2n+1)\pi$, has a maximum near the midspan of the string. However, in all cases examined the maximum of $\langle w_\tau{}^2(x,\tau)\rangle$ over all τ and $0 \leqq x \leqq 1$ always occurs at $x = 1$.

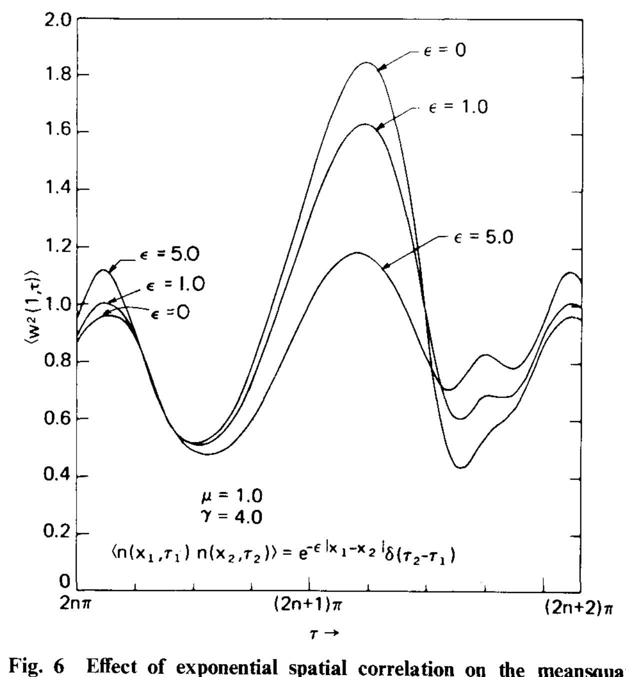

Fig. 6 Effect of exponential spatial correlation on the meansquare displacement at the blade tip.

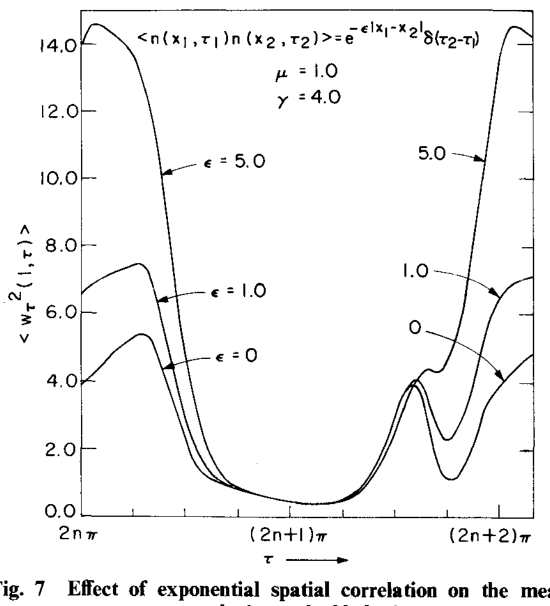

Fig. 7 Effect of exponential spatial correlation on the meansquare velocity at the blade tip.

The effect of the Lock number γ was also investigated. From Table 1, we see that $\langle w^2(1,\tau)\rangle_{max}$ increases much more rapidly with γ than the corresponding rigid blade solution obtained in Ref. 8. They showed that, just as the $\mu = 0$ case,[12] the rigid blade solution may not be an adequate approximation of the actual blade behavior for $\gamma \gtrsim 6$ if $EI/ml^4\Omega^2 \ll 1$.

Table 1 The effect of γ on the meansquare properties

γ	2.0	4.0	6.0	8.0	10.0
$\langle w^2\rangle_{max}$	1.06	2.33	6.98	4.95×10	2.69×10^2
$\langle w^2\rangle_{max}$ [a]	...	2.5	5.80	8.40	...
$\langle w_\tau^{\ 2}\rangle_{max}$	2.40	1.14×10	5.82×10	2.84×10^2	1.20×10^3

[a] Taken from the graph in Ref. 8.

To get some idea of how the spatial variation of the inflow modifies the results obtained so far, we plotted $\langle w^2(1,\tau)\rangle$ and $\langle w_\tau^{\ 2}(1,\tau)\rangle$ for $\varepsilon = 0$, 1.0 and 5.0 with $\mu = 1.0$ and $\gamma = 4$ (see Figs. 6 and 7). We see from these curves that shortening the correlation length tends to decrease the (primary) peak value of $\langle w^2(1,\tau)\rangle$ and increases the peak value of $\langle w_\tau^{\ 2}(1,\tau)\rangle$. $\langle w^2(1,\tau)\rangle_{max}$ for the case $\varepsilon = 5.0$ is only $\frac{2}{3}$ of that for $\varepsilon = 0$ while $\langle w_\tau^{\ 2}(1,\tau)\rangle_{max}$ for $\varepsilon = 5.0$ is almost 3 times that for $\varepsilon = 0$. Increasing ε also moves the location of $\langle w_\tau^{\ 2}\rangle_{max}$ toward the beginning of the forward stroke. The secondary peak value of $\langle w^2(1,\tau)\rangle$ increases with ε and eventually becomes the more dominant of the two peaks. On the other hand, the location of these two peaks does not seem to be significantly affected by the change in ε. Since the actual spatial dependence of $n(x,\tau)$, i.e., the proper form of $R_S(x,y)$, is not fully understood, these observations concerning the effect of ε should be viewed only as a step toward a better understanding of the qualitative behavior of a flexible blade with random loading varying along the blade span. However, it is a simple matter to modify our computer code if a different form of $R_S(x,y)$ is more appropriate.

5. Class of Temporally Correlated Excitations

If the excitation is temporally correlated, we can no longer obtain $p(x,y,\tau)$ and $q(x,y,\tau)$ explicitly as in Sec. 3. The strategy in this case is to try to reduce the problem to one with a temporally uncorrelated excitation by associating $n(x,\tau)$ with the response of some (fictitious) linear dynamical system to a temporally uncorrelated excitation $\hat{n}(x,\tau)$. By this, we mean $n(x,\tau)$ and the steady-state output of the supplementary dynamical system have the same (first- and) second-order statistics. Treating $n(x,\tau)$ as an unknown, the original problem and the supplementary problem together form a new problem with a temporally uncorrelated excitation. While this combined problem is in principle already treated in Sec. 3, a direct application of the method developed there is not recommended since, depending on the supplementary system needed for the reduction, the augmented problem may give rise to a large number of SCF, and the associated computation becomes unnecessarily time consuming. Instead, a less direct method should be used for the augmented problem. This less direct method, which keeps the machine computation essentially at the same level as that for a temporally uncorrelated excitation, will be developed here for a class of $n(x,\tau)$ pertinent to the rotor blade problem. A corresponding development for more general random excitations can be found in Refs. 5 and 14.

We consider in this section a class of zero mean random inflow ratios $n(x,\tau)$ with

$$\langle n(x_1,\tau_1)n(x_2,\tau_2)\rangle = R_S(x_1,x_2)e^{-\alpha|\tau_2-\tau_1|} \tag{17}$$

where α is a positive constant and $R_S(x_2,x_1) = R_S(x_1,x_2)$. For the purpose of obtaining the second-order statistics of $w(x,\tau)$, we may think of $n(x,\tau)$ as the steady-state stationary response of

$$n_\tau + \alpha n = (2\alpha)^{1/2}\hat{n}(x,\tau) \tag{18}$$

where $\hat{n}(x,\tau)$ is a zero mean temporally uncorrelated random function with $\langle \hat{n}(x_1,\tau_1)\hat{n}(x_2,\tau_2)\rangle = R_S(x_1,x_2)\delta(\tau_2-\tau_1)$. By way of the representation

$$n(x,\tau) = (2\alpha)^{1/2}\int_{-\infty}^{\tau} e^{-\alpha(\tau-\xi)}\hat{n}(x,\xi)\,d\xi \tag{19}$$

it is not difficult to show that the autocorrelation function of the steady-state solution of Eq. (18) is the same as that given by Eq. (17) and that

$$\langle n(x_2,\tau_2)\hat{n}(x_1,\tau_1)\rangle = H(\tau_2-\tau_1)(2\alpha)^{1/2}e^{-\alpha(\tau_2-\tau_1)}R_S(x_1,x_2) \tag{20}$$

where $H(z)$ is the unit step function with $H(0) = \frac{1}{2}$. Furthermore, we have from Eq. (19) and the Green's function representation of $w(x,\tau)$ that (see Refs. 5 and 14)

$$\langle w(x_2,\tau)\hat{n}(x_1,\tau)\rangle = \langle w_\tau(x_2,\tau)\hat{n}(x_1,\tau)\rangle = 0 \tag{21}$$

We are now ready to obtain the yet unknown functions

$$p(x,y,\tau) \equiv \langle n(x,\tau)w(y,\tau)\rangle, \quad q(x,y,\tau) \equiv \langle n(x,\tau)w_\tau(y,\tau)\rangle \tag{22}$$

needed in Eqs. (9–11). We do this by formulating a nonstochastic initial-value problem for p and q. To get two equations for these two quantities, we observe that

$$\begin{aligned} p_\tau(x,y,\tau) &= \langle n_\tau(x,\tau)w(y,\tau)\rangle + \langle n(x,\tau)w_\tau(y,\tau)\rangle \\ &= (2\alpha)^{1/2}\langle \hat{n}(x,\tau)w(y,\tau)\rangle - \alpha\langle n(x,\tau)w(y,\tau)\rangle + q(x,y,\tau) \end{aligned} \tag{23}$$

where we have used the ODE [Eq. (18)] to eliminate n_τ. The first term on the right of Eq. (23) vanishes in view of Eq. (21) and what remains may be written as one equation for p and q

$$p_\tau + \alpha p - q = 0 \tag{24}$$

With the help of Eqs. (5) and (21), the identity

$$q_\tau(x,y,\tau) = \langle n_\tau(x,\tau)w_\tau(y,\tau)\rangle + \langle n(x,\tau)w_{\tau\tau}(y,\tau)\rangle \tag{25}$$

may be written as

$$q_\tau + \alpha q - L_{y\tau}[p] + \gamma_0|y + \mu\sin\tau|q = \gamma_0|y + \mu\sin\tau|R_S(x,y) \tag{26}$$

The two coupled PDE Eq. (24) and (26) are supplemented by the initial conditions

$$p(x,y,0) = q(x,y,0) = 0 \tag{27}$$

which follow from Eq. (3) and the boundary conditions

$$p(x,0,\tau) = 0, \quad \lim_{y\to 1}[(1-y^2)p_y] = 0 \tag{28}$$

Together, they determine p and q completely. Note that no boundary condition is needed for q. The initial-value problem may be solved numerically again by a modified explicit scheme similar to that described in Appendix B. Having obtained p and q, we can then solve for the four SCF u, s, t, and v as in Sec. 3.

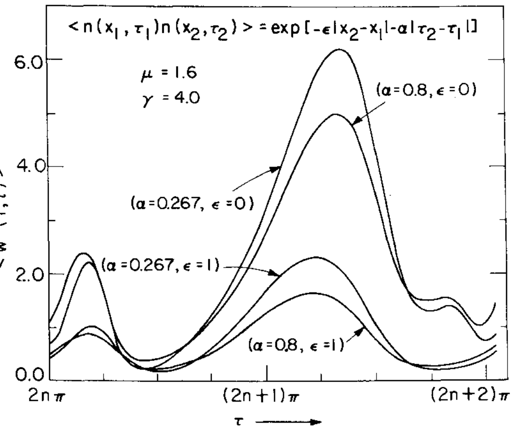

Fig. 8 Steady-state meansquare displacement at the blade tip for an exponentially correlated inflow.

As far as a numerical solution for the present case is concerned, it is more efficient to solve for all six unknowns simultaneously. At the kth time step, the known values for $P^k_{m,n} \equiv p(x_m, y_n, \tau_k)$ and $Q^k_{m,n} \equiv q(x_m, y_n, \tau_k)$ may be used to determine $P^{k+1}_{m,n}$ and $Q^{k+1}_{m,n}$ as well as $U^{k+1}_{m,n}, \ldots, V^{k+1}_{m,n}$. It is not necessary to store $P^k_{m,n}$ and $Q^k_{m,n}$ in core after that. With the same grid over the unit square, the additional amount of computing time required for the supplementary problem is only about 30% of the time required for the solution of SCF. For example, it takes less than 7 min on a UNIVAC 1106 to compute $P^k_{m,n}, Q^k_{m,n}, U^k_{m,n}, \ldots, V^k_{m,n}$ over five blade revolutions with a 21×21 grid and $\Delta = \pi/80$.

6. Steady-State Meansquare Response to a Class of Temporally Correlated Excitations

The solution scheme developed in Sec. 5 has been used to calculate the SCF of the rotor string problem modelled by Eqs. (3) and (5–7) for the case

$$\langle n(x_1, \tau_1) n(x_2, \tau_2) \rangle = \exp[-\varepsilon|x_2 - x_1| - \alpha|\tau_2 - \tau_1|] \quad (29)$$

$0 \leqq x_1, x_2 \leqq 1$. With $\varepsilon = 0$, this particular class of random functions seems to adequately describe the random inflow associated with atmospheric turbulence at altitudes higher than 300 ft above terrain if the effects of the spatial variation of the vertical turbulence component, of the longitudinal turbulence component itself, and of the blade motion are neglected (see Ref. 7 and references therein). In that case, we have $\alpha = 2\mu l/L$ where $L/2$ is the scale length of the vertical turbulence component. L is about 400 ft for an altitude of 300–700 ft above terrain and is several thousand feet for high altitudes. Some of the results of our calculations will be discussed in this section. As in Sec. 4, we are again interested here in 1) the difference between the steady-state meansquare response of a rotor string and a rigid blade for the case $\varepsilon = 0$; and 2) the effect of the dimensionless correlation length ε^{-1} which characterizes the spatial variation of our particular class of random inflows. In addition, we will also examine how a change in the dimensionless correlation time, $1/\alpha$ affects the steady-state nonstationary meansquare blade response, which, for a fixed x, is again a periodic function of τ with a period 2π. In particular, we expect that, for fixed ε, γ, and μ, the steady-state meansquare displacement and velocity (normalized by a multiplicative factor $\alpha/2$) should tend to the corresponding quantities obtained in Sec. 4 for temporally uncorrelated excitations as $\alpha \to \infty$, and they do. As another indication of the accuracy and reliability of our results, we checked and found that these meansquare quantities in the case $\mu = 0$ agree to at least four significant figures with those obtained in Refs. 12 and 15 by completely different methods.

For $\mu = \varepsilon = 0$, it was found in Ref. 12 that there is a 20% discrepancy in the meansquare displacement between a hinged rigid blade and a rotor string for small α when $\gamma = 4$,§ as a substantial amount of the energy in the blade is distributed among the "flexural modes." This difference diminishes as α increases and thus we get a good agreement in the limiting case of a temporally uncorrelated excitation. Therefore, for fixed ε, γ, and $\mu > 0$, we expect the rigid blade solution for our present problem to be substantially different from the string solution for small α. Nevertheless, it is still surprising to see that $\langle w^2(1,\tau)\rangle_{max}$ for the case $\varepsilon = 0$, $\gamma = 4$, $\mu = 1.6$, and $\alpha = 4/15$ $(L/l = 12)$ (see Fig. 8) is less than half of the corresponding peak value for a (hinged) rigid blade obtained in Ref. 7. (The latter is independently confirmed by our own calculation in connection with Ref. 9.) A more extensive numerical study of the effect of μ and α on this discrepancy in $\langle w^2(1,\tau)\rangle_{max}$ revealed the following

1) For a fixed μ, the discrepancy between the two solutions diminishes with increasing α, approaching the small difference for a temporally uncorrelated inflow.

2) For a fixed α, the rigid blade solution for $\langle w^2(1,\tau)\rangle_{max}$ is smaller than the string solution when $\mu = 0$, the difference being about 27% of the string blade value. The discrepancy first diminishes with increasing μ until the relative magnitude of the peak values reverses. Further increases in μ now give rise to increasing difference between the two values.

It appears from these observations that for $\mu > 0$, the negative spring force effect already unduly accentuated in a rigid blade model is intensified by the narrow band temporal behavior of $n(x,\tau)$ when α is small. We expect therefore that for a given stochastic loading with a fixed small α, a rigid blade restrained elastically at its root more adequately approximates the flexible blade since the elastic root restraint reduces or removes the unduly accentuated negative spring rate. From the results given in Ref. 7, we see that for the case $\varepsilon = 0$, $\gamma = 4$, $\mu = 1.6$, and $\alpha = 4/15$, an elastically restrained rigid blade with a restrain parameter value $P = 1.3$ has a peak value of $\langle w^2(1,\tau)\rangle$ which is within only 16% of (and now *smaller* than) the string solution. And a part of this discrepancy is due to a tip loss factor of 0.97 used in Ref. 7.

7. Conclusions

The results obtained by the SCF method brought out a number of important features concerning the nonstationary steady-state behavior of rotor strings subject to random load. For temporally uncorrelated loads, they show that 1) for *very flexible rotor blades* with $EI/\Omega^2 l^4 m \ll 1$, a rigid blade solution may not be adequate for $\gamma \gtrsim 6$ even if we are only interested in the steady-state meansquare displacement and velocity at the blade tip and not their distribution along the blade span, and 2) just as what was found in Ref. 13 for rigid blades, the spatial variation of the random loading cannot be ignored if the correlation length of the excitation is much less than or of the same order of magnitude as the blade length. For an exponentially correlated excitation [see Eq. (29)], the rigid flapping solution is almost always inadequate for *very flexible blades* unless the relaxation time is short so that $\alpha \gg \max(2\pi, \mu)$. Though a model of a rigid blade with a suitable elastic restraint at its root could be used to obtain a reasonably accurate prediction of meansquare displacement and velocity at the blade tip, the appropriate value of the (fictitious) elastic restraint parameter P cannot be determined once for all since it varies with α and μ. Therefore, a flexible blade solution with $EI \neq 0$ is necessary in most cases for temporally correlated excitations.

On the other hand, the response statistics at the blade tip based on a blade model which neglects the bending stiffness of the blade (as we did herein) are in general conservative. Since the blade stiffness due to the centrifugal force is negligibly small near the blade tip and vanishes at the tip, a nonvanishing blade bending stiffness factor, however small, is important in the vicinity of the tip and should help to reduce the blade motion. The following Table 2 taken from Ref. 15 should give some

§ The excitation in this case is a randomly changing collective pitch.

Table 2 The variation of tip mean square response with effective bending stiffness

$EI/ml^4\Omega^2$	$\langle w^2(1,\tau)\rangle$			$\langle w_\tau^2(1,\tau)\rangle$		
	$\varepsilon = 0.0$	$\varepsilon = 1.0$	$\varepsilon = 10.0$	$\varepsilon = 0.0$	$\varepsilon = 1.0$	$\varepsilon = 10$
0	0.2593	0.2322	0.1255	0.3003	0.3822	0.6369
10^{-2}	0.2547	0.2199	0.0945	0.2855	0.3107	0.3763
1.0	0.2504	0.2118	0.0835	0.2796	0.3024	0.3675
10^2	0.2503	0.2116	0.0832	0.2795	0.3023	0.3674

indication of the effect of the bending stiffness factor $EI/ml^4\Omega^2$. The results given in Table 2 are for the case $\mu = 0$ and $\gamma = 4$; *the excitation is due to a randomly changing collective pitch* with an autocorrelation function of the form Eq. (16). For this case, the steady-state response is stationary and the meansquare displacement and velocity at the blade tip are constant in time. From Table 2, we see that for $\mu = 0$, the error associated with setting $\zeta^4 = 0$ is tolerable only for excitation with a correlation length long compared to the blade length. The effect of the blade bending stiffness for $\mu > 0$ and for temporally correlated excitations will be reported elsewhere.

Finally, we mention here that for a given set of γ, μ, and ε, and a 21×21 grid over the unit square, it takes only about 5 min on a UNIVAC 1106 to compute all four SCF over 5 blade revolutions in the case of a temporally uncorrelated excitation. The UNIVAC 1106 being a relatively slow machine, this computing effort is nearly the same as that required by Ref. 7 for a rigid blade. For an exponentially correlated $n(x,\tau)$ [as given by Eq. (29)], it takes less than 7 min to do the same. If the response process is gaussian, then the meansquare statistics obtained are sufficient for the purpose of obtaining design information such as the rate of threshold crossings, etc. In view of its efficiency in generating these meansquare statistics, the SCF method should be very useful for problems of deformable bodies with random loading. If the response is not gaussian, we may use the SCF to compute the autocorrelation of the response by the efficient method outlined in Appendix A.

Appendix A: Autocorrelation

The second-order statistics of the response of flexible structures to random excitation is characterized by the autocorrelation function of the response process. For the rotor blade problem, we mentioned earlier that the autocorrelation function is $r(x,\tau;x_0,\tau_0) = \langle w(x,\tau)w(x_0,\tau_0)\rangle$, $0 \leqq x, x_0 \leqq 1$ and $\tau, \tau_0 \geqq 0$. Note that

$$r(x_0,\tau_0;x,\tau) = r(x,\tau;x_0,\tau_0) \qquad \text{(A1a)}$$

$$r(x,\tau_0;x_0,\tau_0) = u(x,x_0,\tau_0) \qquad \text{(A1b)}$$

As a function of x and τ with x_0 and τ_0 as parameters, $r(x,\tau;x_0,\tau_0)$ satisfies the same PDE as $w(x,\tau)$ except for the right-hand member. More specifically, we have for the string model

$$r_{\tau\tau} + \gamma_0|x+\mu\sin\tau|r_\tau - L_{x\tau}[r] = \gamma_0|x+\mu\sin\tau|c(x,\tau;x_0,\tau_0) \qquad \text{(A2)}$$

where $c(x,\tau;x_0,\tau_0) = \langle n(x,\tau)w(x_0,\tau_0)\rangle$. Equation (A2) is obtained by multiplying Eq. (5) through by $w(x_0,\tau_0)$ and ensemble averaging the resulting equation. From Eqs. (3) and (7), we have also

$$r(0,\tau;x_0,\tau_0) = 0, \quad \lim_{x\to 1}[(1-x^2)r_x(x,\tau;x_0,\tau_0)] = 0 \qquad \text{(A3)}$$

and

$$r(x,0;x_0,\tau_0) = r_\tau(x,0;x_0,\tau_0) = 0 \qquad \text{(A4)}$$

If $c(x,\tau;x_0,\tau_0)$ is known, then Eqs. (A2–A4) determine $r(x,\tau;x_0,\tau_0)$. However, c involves the unknown $w(x_0,\tau_0)$ and must be found by a separate analysis. We can find c by observing that, as a function of x_0 and τ_0 (with x and τ as parameters), it satisfies the PDE

$$c_{\tau_0\tau_0} + \gamma_0|x_0+\mu\sin\tau_0|c_{\tau_0} - L_{x_0\tau_0}[c] = \gamma_0|x_0+\mu\sin\tau_0|\langle n(x_0,\tau_0)n(x,\tau)\rangle \qquad \text{(A5)}$$

the initial conditions

$$c(x,\tau;x_0,0) = c_{\tau_0}(x,\tau;x_0,0) = 0 \qquad \text{(A6)}$$

and the boundary conditions

$$c(x,\tau;0,\tau_0) = 0, \lim_{x_0\to 1}[(1-x_0^2)c_{x_0}(x,\tau;x_0,\tau_0)] = 0 \qquad \text{(A7)}$$

Equations (A5–A7) are obtained in a manner similar to Eqs. (A2–A4). Since the right-hand side of Eq. (A5) is given, we can solve Eqs. (A5–A7) to get c with x and τ as parameters. The solution is then used in Eqs. (A2) to get r. This so-called "autocorrelation function method" is in principle straightforward but is not a practical method whenever we have to settle for a numerical solution for c and r. For a fixed pair of (x_0,τ_0), say (X_0, T_0), to get r for all $0 < \tau \leqq T_0$ and $0 \leqq x \leqq 1$ would require $c(x_m,\tau_k;X_0,T_0)$ for all grid point x_m and time step $\tau_k \leqq T_0$. To get $c(x_m,\tau_k;X_0,T_0)$ for a single pair of (x_m,τ_k), we would have to solve the problem for c once for all $\tau_0 \leqq T_0$. It would take over a day of machine calculation on a UNIVAC 1106 to get $r(x,\tau;x_0,10\pi)$ for the rotor blade problem with a 21-point mesh over the interval [0, 1]. Surely, a more efficient method is desirable. We will describe in what follows a much more efficient method which takes advantage of the fact that we have already found the SCF of the response.

Temporally Uncorrelated Excitations

For this case, we have

$$c(x,\tau;x_0,\tau_0) = 0 \qquad (\tau > \tau_0) \qquad \text{(A8)}$$

This is not surprising as the response at present cannot be correlated with loading in the future since the loading itself is not. The condition (A8) can in fact be proved.[5,14] Since $u(x,x_0,\tau_0)$ and $t(x,x_0,\tau_0)$ have already been obtained in Sec. 3, the PDE Eq. (A2) with the condition (A8), the boundary conditions (A3) and the "initial conditions" (A1b) and

$$r_\tau(x,\tau_0;x_0,\tau_0) = t(x,x_0,\tau_0) \qquad \text{(A9)}$$

determine $r(x,\tau;x_0,\tau_0)$ for $\tau > \tau_0$. We then get r for $\tau < \tau_0$ from the symmetry condition (A1a).

For a fixed pair of (x_0,τ_0), it takes only a few seconds on a UNIVAC 1106 to compute $r(x,\tau;x_0,\tau_0)$ for $0 \leqq x \leqq 1$ and $\tau_0 < \tau \leqq \tau_0 + 6\pi$. The string response is effectively uncorrelated beyond 3 blade revolution for all realistic values of γ.

A Class of Temporally Correlated Excitations

Since the useful condition (A8) no longer holds for temporally correlated excitations, we will have to determine c for $\tau > \tau_0$ by some efficient method. It is not surprising that this can be accomplished by way of a supplementary problem similar to that of Sec. 5. For the class of excitations defined by Eq. (17), again considered to be the steady-state solution of Eq. (19), we have from Eq. (18)

$$c_\tau + \alpha c = 0 \qquad (\tau > \tau_0) \qquad \text{(A10)}$$

for $c(x,\tau;x_0,\tau_0)$ since we have $\langle \hat{n}(x,\tau)w(x_0,\tau_0)\rangle = 0$ for $\tau > \tau_0$ in the case of a temporally uncorrelated $\hat{n}(x,\tau)$. From Eq. (22), we have also

$$c(x,\tau_0;x_0,\tau_0) = \langle n(x,\tau_0)w(x_0,\tau_0)\rangle = p(x,x_0,\tau_0) \qquad \text{(A11)}$$

and therefore,

$$c(x,\tau;x_0,\tau_0) = p(x,x_0,\tau_0)e^{-\alpha(\tau-\tau_0)} \qquad (\tau \geqq \tau_0) \qquad \text{(A12)}$$

With Eq. (A12), the initial-value problem Eqs. (A2, A1b, and A9) determines $r(x,\tau;x_0,\tau_0)$ for $\tau \geqq \tau_0$. The symmetry condition (A1a) gives r for $\tau < \tau_0$.

It is not difficult to see that a similar procedure is available for an efficient determination of $c(x,\tau;x_0,\tau_0)$, $\tau > \tau_0$, whenever $n(x,t)$, at each x, is a more general filtered shot noise.

Appendix B: Modified Explicit Finite-Difference Scheme

A straightforward explicit-difference scheme for the initial value problem Eqs. (8–13) and Eq. (15) can be formulated by using a first-order forward difference formula with a time step Δ

for the time derivatives and second-order central difference formulas with the same mesh spacing h in both x and y for spatial derivatives. At a boundary point along $x = 1$ and $y = 1$, second-order backward difference formulas are used instead. For such an explicit scheme, we have as a finite difference analog of the four PDE

$$
\begin{aligned}
U_{m,n}^{k+1} &= U_{m,n}^k + \Delta(S_{m,n}^k + T_{m,n}^k) \\
T_{m,n}^{k+1} &= T_{m,n}^k + \frac{\Delta}{h^2}[a_m^k U_{m+1,n}^k + b_m^k U_{m,n}^k + c_m^k U_{m-1,n}^k \\
&\quad + h^2(V_{m,n}^k - \gamma_0 |x_m + \mu \sin \tau_k| T_{m,n}^k)] \\
S_{m,n}^{k+1} &= S_{m,n}^k + \frac{\Delta}{h^2}[a_n^k U_{m,n+1}^k + b_n^k U_{m,n}^k + c_n^k U_{m,n-1}^k \\
&\quad + h^2(V_{m,n}^k - \gamma_0 |y_n + \mu \sin \tau_k| S_{m,n}^k)] \\
V_{m,n}^{k+1} &= V_{m,n}^k + \frac{\Delta}{h^2}[a_m^k S_{m+1,n}^k + b_m^k S_{m,n}^k + c_m^k S_{m-1,n}^k \\
&\quad + a_n^k T_{m,n+1}^k + b_n^k T_{m,n}^k + c_n^k T_{m,n-1}^k \\
&\quad - h^2 \gamma_0 (|x_m + \mu \sin \tau_k| \\
&\quad + |y_n + \mu \sin \tau_k|) V_{m,n}^k + h^2 g_{m,n}^k]
\end{aligned} \tag{B1}
$$

for $m, n = 1, 2, \ldots, M-1$ and $k = 0, 1, 2, \ldots$, where Δ and $h = 1/M$ are the time step and mesh spacing (taken to be the same in both the x and y direction), respectively. $U_{m,n}^k, \ldots, V_{m,n}^k$ are the finite-difference solutions for $u(x_m, y_n, \tau_k), \ldots, v(x_m, y_n, \tau_k)$, respectively, and

$$
\begin{aligned}
b_j^k &= -(1 - z_j^2), \quad (x_j, y_j, z_j) = jh, \quad \tau_k = k\Delta \\
(a_j^k, c_j^k) &= \frac{1}{2}(1 - z_j^2) - \left(\frac{h}{2}, -\frac{h}{2}\right)(z_j + \gamma_0 \mu \cos \tau_k |z_j + \mu \sin \tau_k|)
\end{aligned} \tag{B2}
$$

Slightly different formulas for the boundary points along $x = 1$ and along $y = 1$ will not be written out. With $U_{m,n}^0 = \ldots = V_{m,n}^0 = 0$, $U_{m,n}^k, \ldots, V_{m,n}^k$, $k = 1, 2, \ldots$ may be calculated successively by (B1) and the omitted formulas for points along $x = 1$ and $y = 1$. But as we pointed out in Sec. 3, this explicit scheme is inefficient since we need $\Delta = 0(h^2)$ for numerical stability. In view of the time-varying coefficients of the PDE Eqs. (8–11), an implicit scheme does not improve the situation significantly.

On the other hand, Eq. (8) does not involve any spatial derivative. This suggests that the system of four partial differential equations may behave as a hyperbolic system as far as numerical stability is concerned, and the use of an implicit scheme may be avoided even for $\Delta = O(h)$. The question is, how do we modify the explicit scheme Eq. (B1) to get numerical stability for $\Delta = O(h)$ while the modified scheme remains an explicit scheme? One rather obvious modification suggests itself. At the kth step, we may use $U_{m,n}^{k+1}$, already computed from the first equation of (B1), on the right side of the second and third equation instead of $U_{m,n}^k$. These two equations give us $S_{m,n}^{k+1}$ and $T_{m,n}^{k+1}$ which can be used on the right side of the last equation instead of $S_{m,n}^k$ and $T_{m,n}^k$.

Our analysis of a few model problems indicates that there are good reasons for this modified explicit scheme to be stable with $\Delta = O(h)$. For example, the modified explicit scheme is a consistent difference analog of the one-dimensional wave equation written as $u_\tau = v$, $v_\tau = u_{xx}$. On the other hand, the straightforward explicit scheme gives a set of difference equations which is a consistent analog of $u_{\tau\tau} = (1 - \Delta\partial/\partial\tau)u_{xx}$. Also, a Fourier analysis of the modified explicit scheme applied to the system

$$
\begin{aligned}
u_\tau &= s + t, & t_\tau &= v + u_{xx} - d_0 t \\
s_\tau &= v + u_{yy} - d_0 s, & v_\tau &= s_{xx} + t_{yy} - 2d_0 v
\end{aligned} \tag{B3}
$$

where d_0 is a positive constant, gives an amplification factor less than unity in modulus at least in the case of identical wave length in both the x and y directions.

References

[1] Sissingh, G. J., "Dynamics of Rotors Operating at High Advance Ratios," *Journal of Helicopter Society*, Vol. 13, 1968, pp. 56–63.

[2] Lin, Y. K., *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York, 1967.

[3] Bolotin, V. V., *Statistical Methods in Structural Mechanics*, Holden-Day, San Francisco, Calif., 1969.

[4] Bolotin, V. V., *Applications of Methods of the Theory of Probability and the Theory of Reliability in the Calculations of Structures*, Publishing Company of Literatures in Structures, Moscow, 1971 (in Russian).

[5] Wan, F. Y. M., "Linear Partial Differential Equations With Random Forcing," *Studies in Applied Mathematics* (continuation of *Journal of Mathematics and Physics*) Vol. 51, 1972, pp. 163–178.

[6] Gaonkar, G. H. and Hohenemser, K. H., "Flapping Response of Lifting Rotor Blades to Atmospheric Turbulence," *Journal of Aircraft*, Vol. 6, No. 6, Nov.–Dec. 1969, pp. 496–503.

[7] Gaonkar, G. H. and Hohenemser, K. H., "Stochastic Properties of Turbulence Excited Rotor Blade Vibrations," *AIAA Journal*, Vol. 9, No. 3, March 1971, pp. 419–424.

[8] Gaonkar, G. H., "Interpolation of Aerodynamic Damping of Lifting Rotors in Forward Flight from Measured Response Variance," *Journal of Sound and Vibration*, Vol. 18, 1971, pp. 381–389.

[9] Wan, F. Y. M. and Lakshmikantham, C., "Rotor Blade Response to Random Loads: A Direct Time-Domain Approach," *AIAA Journal*, Vol. 11, No. 1, Jan. 1973, pp. 24–28.

[10] Lakshmikantham, C. and Joga Rao, C. V., "Response of Helicopter Rotor Blades to Random Loads Near Hover," *Aeronautical Quarterly*, Vol. 23, 1972, pp. 276–284.

[11] Kana, D. D. and Chu, W. H., "The Response of a Model Helicopter Rotor Blade to Random Excitation During Forward Flight," Project 02-1732 Final Rept., Aug. 1972, Southwestern Research Institute, San Antonio, Texas.

[12] Lakshmikantham, C. and Wan, F. Y. M., "An Algebraic Method for Linear Systems with Stationary Excitations," *Journal of Sound and Vibration*, Vol. 20, 1973, pp. 105–120.

[13] Wan, F. Y. M., "Effect of Spanwise Load-Correlation on Rotor Blade Flapping," AIAA Paper 74-418, Las Vegas, Nev., 1974.

[14] Wan, F. Y. M., "A New Method for Linear Dynamical Problems in Continuum Mechanics with Random Loading," presented at the XIIIth International Congress of Theoretical and Applied Mechanics Aug. 1972 (Moscow, USSR); *Studies in Applied Mathematics*, Vol. 52, 1973, pp. 259–276.

[15] Wan, F. Y. M., "An In-Core Finite Difference Method For Separable Boundary Value Problems on a Rectangle," *Studies in Applied Mathematics*, Vol. 52, 1973, pp. 103–113.